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**MAXIMUM LIKELIHOOD EQUILIBRIA OF GAMES
WITH POPULATION UNCERTAINTY**

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Maximum Likelihood Equilibria of Games with Population Uncertainty

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Abstract: In the games with population uncertainty introduced in this paper, the number and identity of the participating players are determined by chance. Games with population uncertainty are shown to include Poisson games and random-player games. The paper focuses on those strategy profiles that are most likely to yield a Nash equilibrium in the game selected by chance. Existence of maximum likelihood equilibria is established under mild topological conditions.

Keywords: Incomplete information, population uncertainty, maximum likelihood equilibria

JEL Classification: C72

1 Introduction

In games with incomplete information as usually studied by game theorists, the characteristics or types of the participating players are possibly subject to uncertainty, but the number of players is common knowledge. Recently, however, Myerson [8,9,10,11] and Milchtaich [7] proposed models for situations — like elections and auctions — in which it may be inappropriate to assume common knowledge of the player set. In such games with population uncertainty, the set of actual players and their preferences are determined by chance according to a commonly known probability measure (a Poisson distribution in Myerson's work, a point process in Milchtaich's paper) and players have to choose their strategies before the player set is revealed.

The equilibrium concepts introduced by Myerson [11] and Milchtaich [7] for their classes of games with population uncertainty are variants of the Nash equilibrium concept based on a suitably defined expected utility function for the players. Alternatively, the present note stresses those strategy profiles that are *most likely* to yield an equilibrium in the game selected by chance. Maximum likelihood equilibria were introduced in Borm *et al.* [2] in a class of Bayesian games. Gilboa and Schmeidler [5] recently provided an axiomatic foundation for rankings according to the likelihood function.

The σ -algebra underlying the chance event that selects the actual game to be played may be too coarse to make the event in which a specific strategy profile yields an equilibrium measurable. A common mathematical approach (also used in a decision theoretic framework; cf. [4]) to assign probabilities to such events is to use the inner measure induced by the probability measure (cf. [6]). Roughly, the inner measure of an event E is the probability of the largest measurable event included in E .

Under mild topological restrictions, an existence result for maximum likelihood equilibria is derived. Since the result establishes the existence of a maximum of the likelihood function, it differs significantly from standard equilibrium existence results that usually rely on a fixed point argument.

The note is organized as follows. Section 2 recalls definitions and results from topology and measure theory. Section 3 contains the definition of games with population uncertainty. These games are shown to include the Poisson games of Myerson [11] and the random-player games of Milchtaich [7]. In Section 4, maximum likelihood equilibria are defined and shown to exist under mild topological restrictions on a game with population uncertainty.

2 Preliminaries

For easy reference, this section summarizes results and definitions from topology and measure theory that are used in the rest of the paper. See [1,6] for additional information.

Let X and Y be topological spaces. A function $f : X \rightarrow Y$ is *sequentially continuous* if for every $x \in X$ and every sequence $(x^n)_{n=1}^\infty$ in X converging to x , it holds that $\lim_n f(x^n) = f(x)$. Sequential continuity is implied by continuity of functions; the converse is not true [1, Theorem

2.25]. A function $f : X \rightarrow \mathbb{R}$ is *sequentially upper semicontinuous* if for every $x \in X$ and every sequence $(x^n)_{n=1}^\infty$ in X converging to x , it holds that $\limsup_n f(x^n) \leq f(x)$. Sequential upper semicontinuity is implied by upper semicontinuity of functions; the converse is not true [1, Lemma 2.40]. A set $A \subseteq X$ is

- *sequentially closed* if for every $x \in X$ and every sequence $(x^n)_{n=1}^\infty$ in A converging to x , it holds that $x \in A$. Every closed set is sequentially closed; the converse is not true [1, Example 2.10].
- *sequentially compact* if every sequence in A has a subsequence converging to an element of A . Every compact set is sequentially compact; the converse is not true [1, Theorem 2.29].

Let (Ω, Σ, P) be a probability space, where Ω is a nonempty set, Σ is a σ -algebra on Ω , and P a probability measure on Σ . The *inner measure* $P_*(E)$ of a set $E \subseteq \Omega$ is defined as

$$P_*(E) := \sup \{P(F) \mid F \in \Sigma, F \subseteq E\}.$$

Roughly speaking, the inner measure of an event E is the probability of the largest measurable event contained in E . $P_*(E)$ is well-defined, since the set $\{P(F) \mid F \in \Sigma, F \subseteq E\}$ is nonempty ($\emptyset \in \Sigma, \emptyset \subseteq E$) and bounded above by one (P is a probability measure). Moreover, $P_*(E) = P(E)$ if $E \in \Sigma$. The *lower integral* of a function $f : \Omega \rightarrow \mathbb{R}$ is defined as

$$\int_* f(\omega) P(d\omega) := \sup \left\{ \int g(\omega) P(d\omega) \mid g \text{ Lebesgue integrable}, g \leq f \right\}. \quad (1)$$

Inner measures and lower integrals are related via the following equality:

$$\forall E \subseteq \Omega : \quad P_*(E) = \int_* 1_E(\omega) P(d\omega), \quad (2)$$

where 1_E is the indicator function for the set E . Clearly, if f is itself Lebesgue integrable, then

$$\int_* f(\omega) P(d\omega) = \int f(\omega) P(d\omega). \quad (3)$$

Below, a version of Fatou's Lemma is shown to hold for lower integrals. First, a lemma is needed.

Lemma 2.1 *Let $f : \Omega \rightarrow \mathbb{R}$ be such that $\int_* f(\omega) P(d\omega) < \infty$. Then there exists a Lebesgue integrable function $h : \Omega \rightarrow \mathbb{R}$ such that $h \leq f$, and $\int_* f(\omega) P(d\omega) = \int h(\omega) P(d\omega)$.*

Proof. By (1) there is a sequence $(h^n)_{n=1}^\infty$ of Lebesgue integrable functions such that $h^n \leq f$ and $\int h^n(\omega) P(d\omega) \geq \int_* f(\omega) P(d\omega) - \frac{1}{n}$ for each $n \in \mathbb{N}$. The Lebesgue integrable function $h := \sup_n h^n$ clearly satisfies $h \leq f$. Consequently, $\int h(\omega) P(d\omega) \leq \int_* f(\omega) P(d\omega)$ and $\int h(\omega) P(d\omega) \geq \int h^n(\omega) P(d\omega) \geq \int_* f(\omega) P(d\omega) - \frac{1}{n}$ for all $n \in \mathbb{N}$, so $\int h(\omega) P(d\omega) \geq \int_* f(\omega) P(d\omega)$. Hence $\int h(\omega) P(d\omega) = \int_* f(\omega) P(d\omega)$. \square

Proposition 2.2 *Let $(f^n)_{n=1}^\infty$ be a sequence of functions $f^n : - \rightarrow \mathbb{R}$ and $g : - \rightarrow \mathbb{R}$ a Lebesgue integrable function such that $f^n \leq g$ for all $n \in \mathbb{N}$. Then*

$$\limsup_n \int_* f^n(\omega) P(d\omega) \leq \int_* \limsup_n f^n(\omega) P(d\omega).$$

Proof. Lemma 2.1 implies that for each $n \in \mathbb{N}$ there exists a Lebesgue integrable function h^n with $h^n \leq f^n \leq g$ such that $\int_* f^n(\omega) P(d\omega) = \int h^n(\omega) P(d\omega)$. To this sequence $(h^n)_{n=1}^\infty$, the classical Fatou Lemma applies:

$$\limsup_n \int_* f^n(\omega) P(d\omega) = \limsup_n \int h^n(\omega) P(d\omega) \leq \int \limsup_n h^n(\omega) P(d\omega). \quad (4)$$

Since $\limsup_n f^n \geq \limsup_n h^n$ and $\limsup_n h^n$ is Lebesgue integrable, it follows from (1) and (3) that

$$\int_* \limsup_n f^n(\omega) P(d\omega) \geq \int_* \limsup_n h^n(\omega) P(d\omega) = \int \limsup_n h^n(\omega) P(d\omega). \quad (5)$$

Combining (4) and (5) yields the desired result. \square

3 Games with Population Uncertainty

In this section, games with population uncertainty are formally defined. Subsequently, games with population uncertainty are briefly compared with the random-player games of Milchtaich [7] and the Poisson games of Myerson [11].

The set of potential players is a nonempty set N . Each potential player $i \in N$ has a nonempty strategy set A_i . The actual player set is determined by chance according to a probability space $(-, \Sigma, P)$. To each state $\omega \in -$ is associated a strategic game $G_\omega = \langle N_\omega, A_\omega, (\succeq_{i,\omega})_{i \in N_\omega} \rangle$ with a nonempty set of actual players $N_\omega \subseteq N$ having strategy space $A_\omega := \times_{i \in N_\omega} A_i$ and each player $i \in N_\omega$ having a preference relation $\succeq_{i,\omega}$ over A_ω . The tuple $\langle N, (A_i)_{i \in N}, -, \Sigma, P, (G_\omega)_{\omega \in -} \rangle$ is a *game with population uncertainty*.

This definition captures the idea that is also present in the work of Myerson [11] and Milchtaich [7] on games with population uncertainty: there is uncertainty about the exact state of nature $\omega \in -$, and consequently about the game $G_\omega = \langle N_\omega, A_\omega, (\succeq_{i,\omega})_{i \in N_\omega} \rangle$ that will be played. Analogous to the related literature, the probability measure P , according to which the state of nature is determined, is assumed to be common knowledge among the potential players.

Some additional notation: $A := \times_{i \in N} A_i$ denotes the collection of strategy profiles of the potential players. Assume the potential players have fixed a strategy profile $a = (a_i)_{i \in N}$. For notational convenience, denote by $a_\omega := (a_i)_{i \in N_\omega}$ the strategy profile of the players engaged in the game $G_\omega = \langle N_\omega, A_\omega, (\succeq_{i,\omega})_{i \in N_\omega} \rangle$ that is played if state $\omega \in -$ is realized. The *best response correspondence* of G_ω is denoted by $BR_\omega : A_\omega \rightrightarrows A_\omega$, i.e.,

$$\forall a_\omega \in A_\omega : BR_\omega(a_\omega) := \times_{i \in N_\omega} \{b_i \in A_i \mid (b_i, (a_\omega)_{-i}) \succeq_{i,\omega} (c_i, (a_\omega)_{-i}) \text{ for all } c_i \in A_i\},$$

where $(a_\omega)_{-i} = (a_j)_{j \in N_\omega \setminus \{i\}}$ denotes the strategy profile of the players in $N_\omega \setminus \{i\}$.

Games with population uncertainty as defined above generalize the Poisson games of Myerson [11] and the random-player games of Milchtaich [7]. Milchtaich [7, p. 5] introduces random-player games as consisting of:

- a compact metric space X of potential players;
- a simple point process (cf. [3]) on X that determines the actual set of players;
- strategy sets defined by means of a continuous function ξ from a compact metric space Y to X . The strategy set of player $i \in X$ equals $\xi^{-1}(\{i\})$;
- bounded and measurable payoff functions giving a payoff $u(s, S)$ to an actual player who plays s when the strategies of the other players are S .

Every random-player game is easily seen to be a game $\langle N, (A_i)_{i \in N}, -, \Sigma, P, (G_\omega)_{\omega \in -} \rangle$ with population uncertainty: set N equal to X , A_i equal to $\xi^{-1}(\{i\})$, identify $(-, \Sigma, P)$ with the distribution of the simple point process, and the preferences with the utility functions u . Milchtaich [7, p.6, Example 3] indicates that the Poisson games of Myerson [11] are random-player games and consequently games with population uncertainty.

4 Maximum Likelihood Equilibria

The equilibrium concepts introduced by Myerson [11] and Milchtaich [7] for their classes of games with population uncertainty are variants of the Nash equilibrium concept based on a suitably defined expected utility function for the players. This section presents an alternative approach by stressing those strategy profiles that are *most likely* to yield a Nash equilibrium in the game selected by chance. Maximum likelihood equilibria were introduced in Borm *et al.* [2] for a class of Bayesian games and were considered more recently in Voorneveld [12]. In this section we define maximum likelihood equilibria for games with population uncertainty and provide an existence result.

Consider a game $\langle N, (A_i)_{i \in N}, -, \Sigma, P, (G_\omega)_{\omega \in -} \rangle$ with population uncertainty. The players in N must plan their strategies in ignorance of the stochastic state of nature ω that is realized. A strategy profile $a = (a_i)_{i \in N} \in A$ gives rise to a Nash equilibrium if the realized state of nature is an element of the set

$$\{\omega \in - \mid a_\omega \text{ is a Nash equilibrium of } G_\omega\} = \{\omega \in - \mid a_\omega \in BR_\omega(a_\omega)\}.$$

How likely is this event? Although this set need not be measurable (i.e., an element of the σ -algebra Σ), a common mathematical approach in such cases is to define its likelihood via its inner measure

$$P_*(\{\omega \in - \mid a_\omega \in BR_\omega(a_\omega)\}),$$

the probability of the largest measurable set of states of nature in which the strategy profile $a = (a_i)_{i \in N}$ gives rise to a Nash equilibrium. See [4] for another paper using inner measures in a decision theoretic framework. Formally, define the *Nash likelihood function* $L : A \rightarrow [0, 1]$ for each $a = (a_i)_{i \in N} \in A$ as

$$L(a) := P_*(\{\omega \in \Omega \mid a_\omega \in BR_\omega(a_\omega)\}), \quad (6)$$

and define $a = (a_i)_{i \in N}$ to be a *maximum likelihood equilibrium* if

$$L(a) = \sup_{b \in A} L(b).$$

In a recent paper, Gilboa and Schmeidler [5] provided an axiomatic foundation for rankings according to the likelihood function. The following theorem provides an existence result for maximum likelihood equilibria.

Theorem 4.1 *Consider a game $\langle N, (A_i)_{i \in N}, -, \Sigma, P, (G_\omega)_{\omega \in \Omega} \rangle$ with population uncertainty. If there are topologies on A and the sets A_ω for each $\omega \in \Omega$ such that*

(α) *A is sequentially compact;*

(β) *for every $\omega \in \Omega$ the graph $\text{gph } BR_\omega := \{(a, b) \in A_\omega \times A_\omega \mid b \in BR_\omega(a)\}$ is sequentially closed in $A_\omega \times A_\omega$;*

(γ) *for every $\omega \in \Omega$ the function from A to A_ω defined by $a \mapsto a_\omega$ is sequentially continuous,*

then the set of maximum likelihood equilibria is nonempty.

Proof. The set $\{L(b) \mid b \in A\}$ is nonempty and bounded above by one. Hence its supremum exists. Let $(a^n)_{n=1}^\infty$ be a sequence in A such that $\lim_n L(a^n) = \sup_{b \in A} L(b)$. Since A is sequentially compact by (α), the sequence $(a^n)_{n=1}^\infty$ has a subsequence converging to an element $a \in A$. Without loss of generality, this subsequence is taken to be $(a^n)_{n=1}^\infty$ itself: $\lim_n a^n = a$. This $a \in A$ is shown to be a maximum likelihood equilibrium.

For each $\omega \in \Omega$ and $b_\omega \in A_\omega$ it holds by definition that $b_\omega \in BR_\omega(b_\omega)$ if and only if $(b_\omega, b_\omega) \in \text{gph } BR_\omega$. Hence

$$\forall \omega \in \Omega, \forall b_\omega \in A_\omega : 1_{\{\omega \in \Omega \mid b_\omega \in BR_\omega(b_\omega)\}}(\omega) = 1_{\text{gph } BR_\omega}(b_\omega, b_\omega). \quad (7)$$

We show that for every $\omega \in \Omega$, the function from A_ω to $\{0, 1\}$ defined by $b_\omega \mapsto 1_{\text{gph } BR_\omega}(b_\omega, b_\omega)$ is sequentially upper semicontinuous. Fix $\omega \in \Omega$ and a sequence $(b_\omega^n)_{n=1}^\infty$ in A_ω converging to $b_\omega \in A_\omega$. To show: $\limsup_n 1_{\text{gph } BR_\omega}(b_\omega^n, b_\omega^n) \leq 1_{\text{gph } BR_\omega}(b_\omega, b_\omega)$. Since $(1_{\text{gph } BR_\omega}(b_\omega^n, b_\omega^n))_{n=1}^\infty$ is a sequence in $\{0, 1\}$, the inequality trivially holds if $1_{\text{gph } BR_\omega}(b_\omega, b_\omega) = 1$. So assume that $1_{\text{gph } BR_\omega}(b_\omega, b_\omega) = 0$. It remains to prove that

$$\limsup_n 1_{\text{gph } BR_\omega}(b_\omega^n, b_\omega^n) = \lim_n (\sup \{1_{\text{gph } BR_\omega}(b_\omega^m, b_\omega^m) \mid m \geq n\}) = 0,$$

i.e., that there exists an $n \in \mathbb{N}$ such that $1_{\text{gph } BR_\omega}(b_\omega^m, b_\omega^m) = 0$ for each $m \geq n$. Suppose, to the contrary, that such an n does not exist. Then there is a subsequence $(1_{\text{gph } BR_\omega}(b_\omega^{n(k)}, b_\omega^{n(k)}))_{k=1}^\infty$ such that $1_{\text{gph } BR_\omega}(b_\omega^{n(k)}, b_\omega^{n(k)}) = 1$ for each $k \in \mathbb{N}$, i.e., $(b_\omega^{n(k)}, b_\omega^{n(k)}) \in \text{gph } BR_\omega$ for each $k \in \mathbb{N}$. Since $\text{gph } BR_\omega$ is sequentially closed by (β) , this implies that $\lim_k (b_\omega^{n(k)}, b_\omega^{n(k)}) = (b_\omega, b_\omega) \in \text{gph } BR_\omega$, contradicting the assumption that $1_{\text{gph } BR_\omega}(b_\omega, b_\omega) = 0$. This settles the preliminary work. In the sequence of (in)equalities below,

- the first equality is (6),
- the second equality follows from (2) and (7),
- the first inequality follows from sequential upper semicontinuity of $b_\omega \mapsto 1_{\text{gph } BR_\omega}(b_\omega, b_\omega)$ and the fact that $a_\omega^n \rightarrow a_\omega$, since $a^n \rightarrow a$ and $a \mapsto a_\omega$ is sequentially continuous by (γ) ,
- the second inequality follows from Fatou's Lemma for lower integrals (Proposition 2.2),
- the third equality follows from (2), (6), and (7),
- the final equality follows from $\sup_{b \in A} L(b) = \lim_n L(a^n) = \limsup_n L(a^n)$.

The following (in)equalities hold:

$$\begin{aligned}
L(a) &= P_*(\{\omega \in - \mid a_\omega \in BR_\omega(a_\omega)\}) \\
&= \int_* 1_{\text{gph } BR_\omega}(a_\omega, a_\omega) P(d\omega) \\
&\geq \int_* \limsup_n 1_{\text{gph } BR_\omega}(a_\omega^n, a_\omega^n) P(d\omega) \\
&\geq \limsup_n \int_* 1_{\text{gph } BR_\omega}(a_\omega^n, a_\omega^n) P(d\omega) \\
&= \limsup_n L(a^n) \\
&= \sup_{b \in A} L(b).
\end{aligned}$$

But then $a \in A$ is a maximum likelihood equilibrium of $\langle N, (A_i)_{i \in N}, -, \Sigma, P, (G_\omega)_{\omega \in -} \rangle$. \square

A compactness condition like (α) is standard in equilibrium existence results. The sequential continuity condition (γ) guarantees that a convergent sequence of strategy profiles in A is projected to a convergent sequence of strategy profiles in the games $(G_\omega)_{\omega \in -}$ that are realized in the different states of nature. This condition is automatically fulfilled if for instance the topologies on A and $(A_\omega)_{\omega \in -}$ are taken to be the product topologies of those on the strategy spaces A_i of the players $i \in N$. The closedness condition (β) on the graphs of best response correspondences $(BR_\omega)_{\omega \in -}$ is closely related to the upper semicontinuity conditions imposed on best response correspondences in equilibrium existence proofs using the Kakutani fixed point theorem. As

a consequence, even though the existence proof of maximum likelihood equilibria significantly differs from existence proofs involving a fixed point argument, the basic conditions driving the result are the same.

References

1. C.D.Aliprantis and K.C.Border, “Infinite Dimensional Analysis; A Hitchhiker’s Guide”, Springer-Verlag, New York, 1994.
2. P.E.M.Borm, R.Cao, and I.García-Jurado, Maximum likelihood equilibria of random games, *Optimization* **35** (1995), 77-84.
3. D.J.Daley and D.Vere-Jones, “An Introduction to the Theory of Point Processes”, Springer-Verlag, New York, 1988.
4. R.Fagin and J.Y.Halpern, Uncertainty, belief, and probability, *Computational Intelligence* **7** (1991), 160-173.
5. I.Gilboa and D.Schmeidler, Inductive inference: an axiomatic approach, mimeo, Tel Aviv University, 1999.
6. P.Halmos, “Measure Theory”, Van Nostrand, New York, 1950.
7. I.Milchtaich, Random-player games, Northwestern University Math Center, discussion paper 1178, 1997.
8. R.B.Myerson, Large Poisson games, Northwestern University Math Center, discussion paper 1189, 1997, to appear in *J. Econ. Theory*.
9. R.B.Myerson, Comparison of scoring rules in Poisson voting games, Northwestern University Math Center, discussion paper 1214, 1998.
10. R.B.Myerson, Extended Poisson games and the Condorcet jury theorem, *Games Econ. Behav.* **25** (1998), 111-131.
11. R.B.Myerson, Population uncertainty and Poisson games, *Int. J. Game Theory* **27** (1998), 375-392.
12. M.Voorneveld, “Potential Games and Interactive Decisions with Multiple Criteria”, Ph.D. thesis, CentER Dissertation Series 61, Tilburg University.